Saturday, 20 February 2021
Triangle Exterior Bisector

Ali Salehson
Gothenburg Sweden

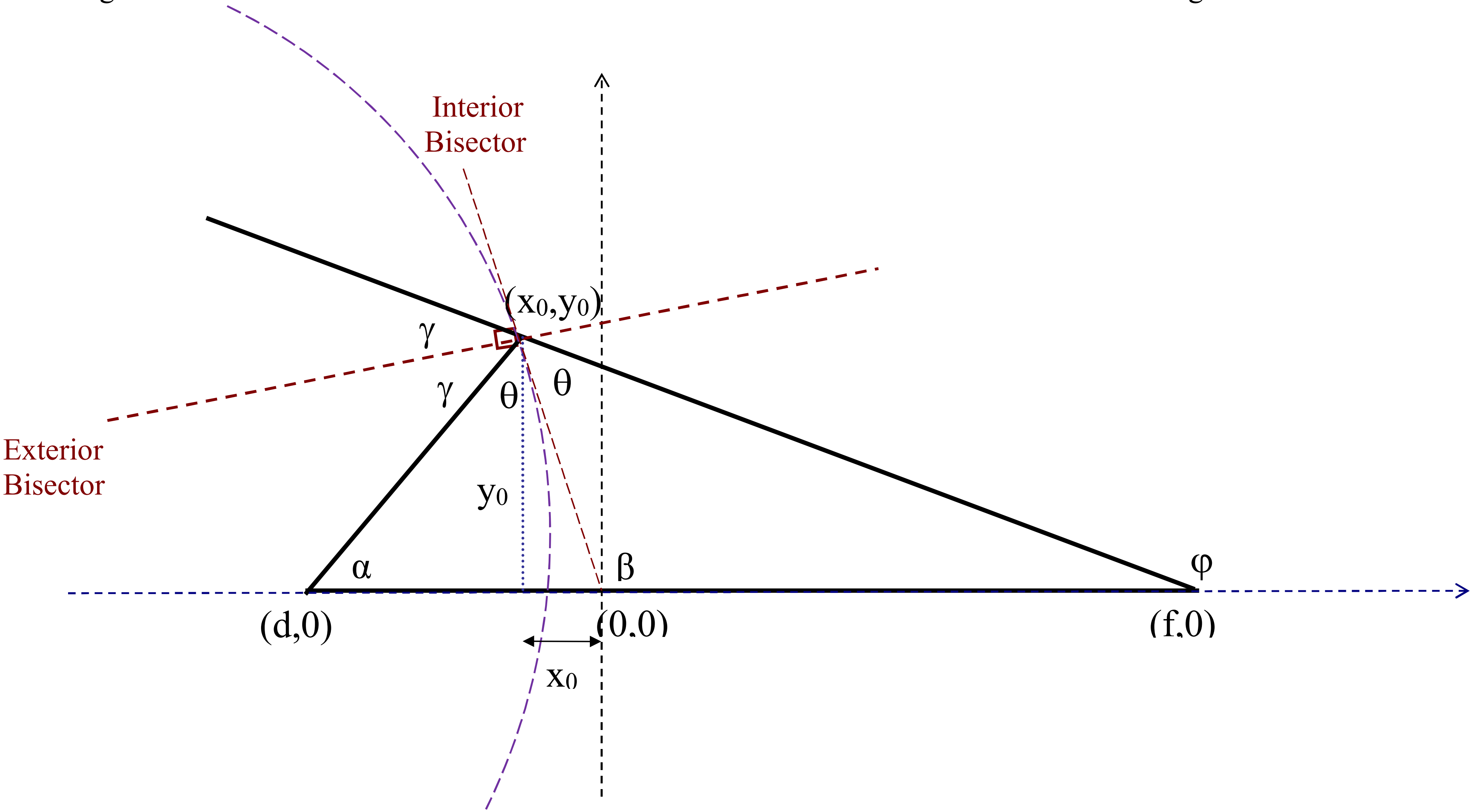

Figure 1: Exterior angle $2\gamma < 90^{o}$

Ali Salehson
Gothenburg Sweden

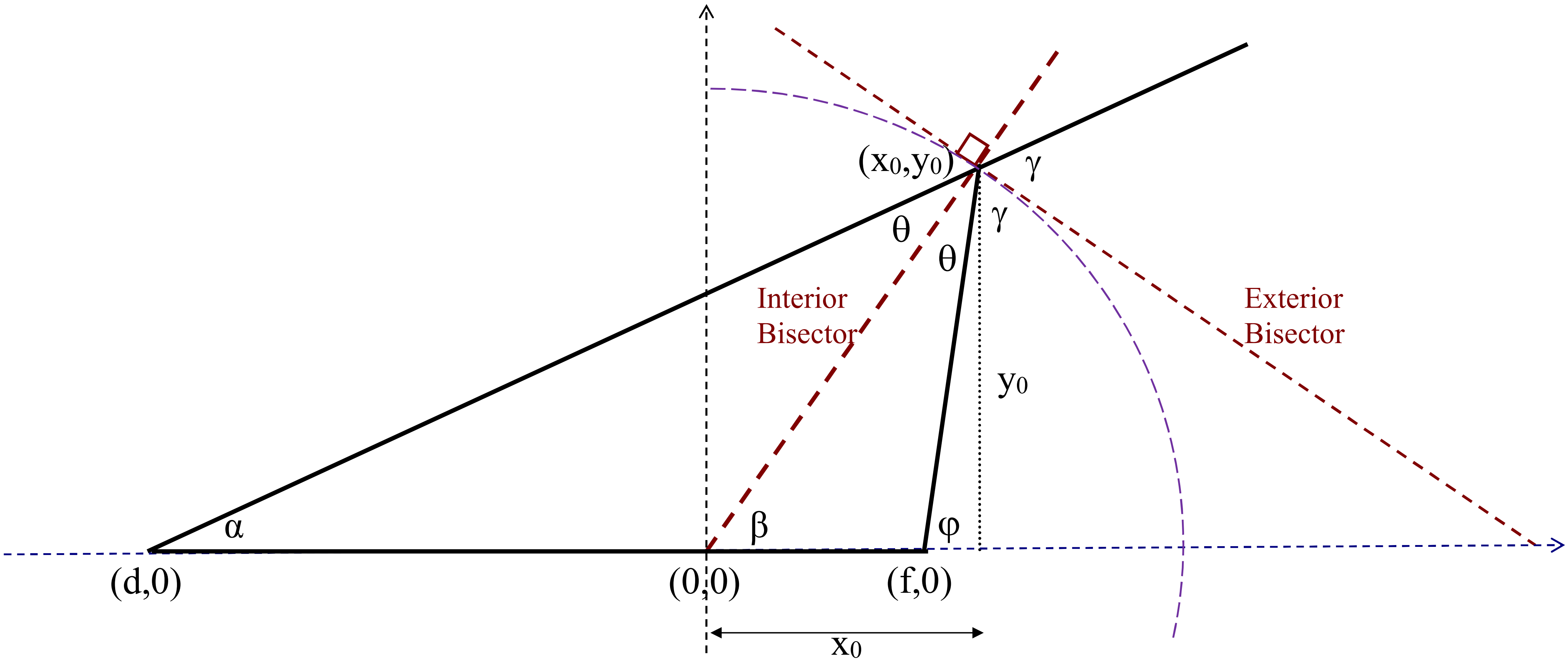

Figure 2: Exterior angle $2\gamma > 90^{o}$

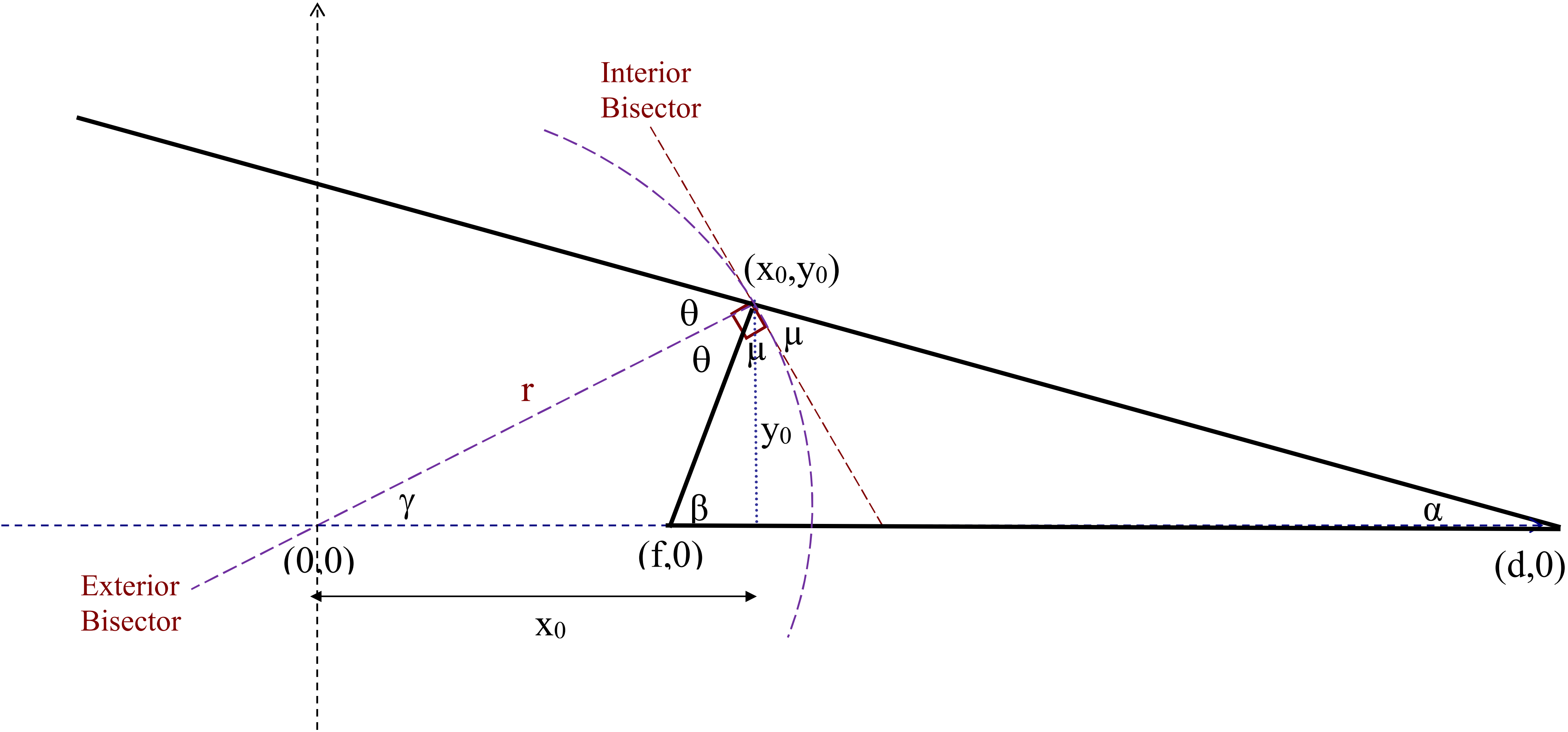

Figure 3: Exterior bisector

Ali Salehson
Gothenburg Sweden

Based on the “salehson theorem” of the triangle **interior** angle bisector, the equation:

$$2\,x_0 / r^2 = 1/f + 1/d$$

can be easily extended and applied to the bisector of the **exterior** angle adjacent to the vertex interior angle. This article is aimed to explicate this extension.

As it has been mentioned in previous articles, the following assumptions are still valid for both the interior and exterior bisectors and should be considered strictly relative to the case presented in the accompanied figure.

- Any triangle with its base (side) is placed on the x-axis of the Cartesian x-y coordinates system, the point (0,0) will be the **reference point** for all length (distance) measurements. This reference point is the point where the angle bisector intersects with the base.

- The base will consist of two **parts** with lengths $|f|$ and $|d|$ due to the intersection of the **interior** bisector. These are actually distances of the vertices on the base side relative the reference point (0,0). The vertices are shown as the points $(f,0)$ and $(d,0)$ respectively i.e. they are obviously located on the x-axis. In the case of the **exterior** bisector I use the more general definition of f and d as being the x-coordinates of the vertices of the base side. This definition is valid for both types of bisector.

- The triangle top vertex (of the interior angle facing the base) is the point (x_0,y_0). The coordinate y_0 is the length of the **altitude**, i.e. the straight line through the top vertex and perpendicular (i.e. forming a right angle with) to the opposite side (the base).

- The length of the bisector line; which is denoted by the letter r, is the length of the straight line between (0,0) and (x_0,y_0). This means; as for a circle: $r^2 = (x_0)^2 + (y_0)^2$
This is why I do include a curve (piece of a circle circumference) in figures.

- Since the sum of the adjacent exterior angle (2θ) and the interior vertex angle (2μ) is well known to be 180° i.e. :

$$(2\theta) + (2\mu) = 180^\circ \text{ and then } \theta + \mu = 90^\circ$$

This means that the two bisectors are perpendicular to each other.

Note: Please note that the top vertex **interior** angle is denoted by (2μ) in Figure 3. Previously this angle has been denoted by (2θ) as shown in Figures 1 and 2. On the other hand the vertex **exterior** angle is denoted now by (2θ) as shown in Figure 3 while previously the angle has been denoted by (2μ) as shown in Figures 1 and 2. *There is reason for this symbol exchanging which will be clearly explained in later articles.*

The inherent exterior angles $\theta = \gamma + \alpha$, and $\beta = \theta + \gamma$, which give:

$\beta = 2\gamma + \alpha$, or $2\gamma = \beta - \alpha$ and therefore $\tan(2\gamma) = \tan(\beta - \alpha)$ is the equation that is used in this article in order to derive the intended equation .

Referring to Figure 3:

$\tan(\gamma) = y_0 / x_0$, $\tan(\beta) = y_0 / (x_0 - f)$, $\tan(\alpha) = y_0 / (d - x_0)$, d here is positive!

It's obvious that the tangents of these angles γ, β and α have the same numerator y_0 but this time θ **half of the exterior angle** is involved.

Considering the equation $\tan(2\gamma) = \tan(\beta - \alpha)$ and by applying the tangent of angles sum :

Ali Salehson
Gothenburg Sweden

$$\frac{2\tan(\gamma)}{1 - \tan^2(\gamma)} = \frac{\tan(\beta) - \tan(\alpha)}{1 + \tan(\beta)\tan(\alpha)}$$

This equation can be simplified by multiplying the numerator of one side with the denominator of the other side which results another equation:

$$[\, 2\{\tan(\gamma)\} \,] \, [\, 1 + \tan(\beta) \tan(\alpha) \,] = [\tan(\beta) - \tan(\alpha)] \, [(1 - \tan^2(\gamma)]$$

Prior continue it is important to be very careful with the ± signs. Now substitute the tangents with their values:

$$[\, 2y_0 / x_0 \,] \, [1 + [y_0^2 / \{(x_0 - f)\, (d - x_0)\}\,]\,] = [\, \{y_0 / (x_0 - f)\} - \{y_0 / (d - x_0)\}\,] \, [\, 1 - (y_0^2 / x_0^2)\,]$$

Now by simply exchange the signs, the above equation can be rewritten:

$$[\, 2y_0 / x_0 \,] \, [1 - [\, y_0^2 / \{(x_0 - f)\, (x_0 - d)\}\,]\,] = [\, \{y_0 / (x_0 - f)\} + \{y_0 / (x_0 - d)\}\,] \, [\, 1 - (y_0^2 / x_0^2)\,]$$

The above equation is exactly the same one that has been found in the calculation of f and d when using the interior bisector. This implies that the final equation will be:

Ali Salehson
Gothenburg Sweden

$$2\, x_0 / r^2 = 1/f + 1/d$$

Here comes a repeated operation:

We can rewrite the last equation in order to recognize common values on both sides:

$$\frac{2y_0}{x_0}\left[1 - \frac{y_0^2}{(x_0 - f)(x_0 - d)}\right] = \left[\frac{y_0}{(x_0 - f)} + \frac{y_0}{(x_0 - d)}\right]\left[1 - \frac{y_0^2}{x_0^2}\right]$$

It can be noticed that y_0 is common numerator on both sides of the equation which can be eliminated by dividing by y_0 and I get:

$$\frac{2}{x_0}\left[1 - \frac{y_0^2}{(x_0 - f)(x_0 - d)}\right] = \left[\frac{1}{(x_0 - f)} + \frac{1}{(x_0 - d)}\right]\left[1 - \frac{y_0^2}{x_0^2}\right]$$

Further simplifying:

$$\frac{2}{x_0}\left[\frac{(x_0 - f)(x_0 - d) - y_0^2}{[(x_0 - f)(x_0 - d)]}\right] = \left[\frac{(x_0 - d) + (x_0 - f)}{[(x_0 - f)(x_0 - d)]}\right]\left[\frac{x_0^2 - y_0^2}{x_0^2}\right]$$

Ali Salehson
Gothenburg Sweden

Multiplying both sides by $x_0 [(x_0 - f) (x_0 - d)]$ will result:

$$2 [x_0^2 - y_0^2 - (f + d) x_0 + f d] = [2 x_0 - (f + d)] \left[\frac{x_0^2 - y_0^2}{x_0} \right]$$

Again multiplying both sides by x_0 the equation becomes:

$$2 (x_0^2 - y_0^2) x_0 - 2 (f + d) x_0^2 + 2 f d x_0 = 2 (x_0^2 - y_0^2) x_0 - (f + d) x_0^2 + (f + d) y_0^2$$

On both sides the term $2 (x_0^2 - y_0^2) x_0$ can be eliminated and the equation becomes:

$$2 f d x_0 = (f + d) x_0^2 + (f + d) y_0^2 = (f + d) (x_0^2 + y_0^2)$$

Since $(x_0^2 + y_0^2)$ is equal to the square of the length of the exterior bisector (the purple dashed line) which is denoted by the letter r, the last equation will become simpler:

$2 f d x_0 = (f + d) r^2$ and it can be rewritten:
$2 x_0 / r^2 = (f + d) / f d$, and finally:

$$2 x_0 / r^2 = 1/f + 1/d$$

Summary and Conclusions:

1. The following equation:

$$2\,x_0 / r^2 = 1/f + 1/d$$

   is valid for all cases and types of triangle angle bisection.

2. This equation is applicable to any single triangle with one side (called the **base**) is placed on the x-axis of the Cartesian xy-coordinate system such that the origin point (0,0) is determined by the **intersection point** between the base line (or its extension) with ***either*** the **interior bisector** of the top triangle vertex angle ***or*** the **exterior bisector** of the adjacent angle to the top triangle vertex angle.

3. The origin point (0,0) is called the **reference point** for the angle bisection.

4. All variables x_0, f and d can be $\pm$ and they are the x-coordinates of points $(x_0,0)$, $(f,0)$ and $(d,0)$ respectively.

5. The point $(x_0,0)$ is the **intersection point** of the **altitude** of the top triangle vertex angle facing the base. The altitude is a straight line perpendicular to the base. The length of the

altitude is denoted by $|y_0|$ or just y_0 when the triangle is placed upsides. With other words the altitude is the line between the **vertex point** (x_0,y_0) and the point $(x_0,0)$.

6. The points $(f,0)$ and $(d,0)$ are the vertex points of the other two triangle angles **relative** the chosen reference point (0,0). This means that f and d are the x-coordinates of the base endpoints. For the case of **interior** bisector, $|f|$ and $|d|$ are actually the lengths of the parts of the base.

7. The bisector line itself is the line between the vertex point (x_0,y_0) and the reference point (0,0). The length of the bisector line is denoted by r and obviously:

$$r^2 = (x_0)^2 + (y_0)^2$$

8. The two bisector lines are perpendicular to each other as it has been explained earlier in the text of this article.

9. For example but not only, when x_0 and r are determined then the relationship between f and d is evidently expressed by the equation.