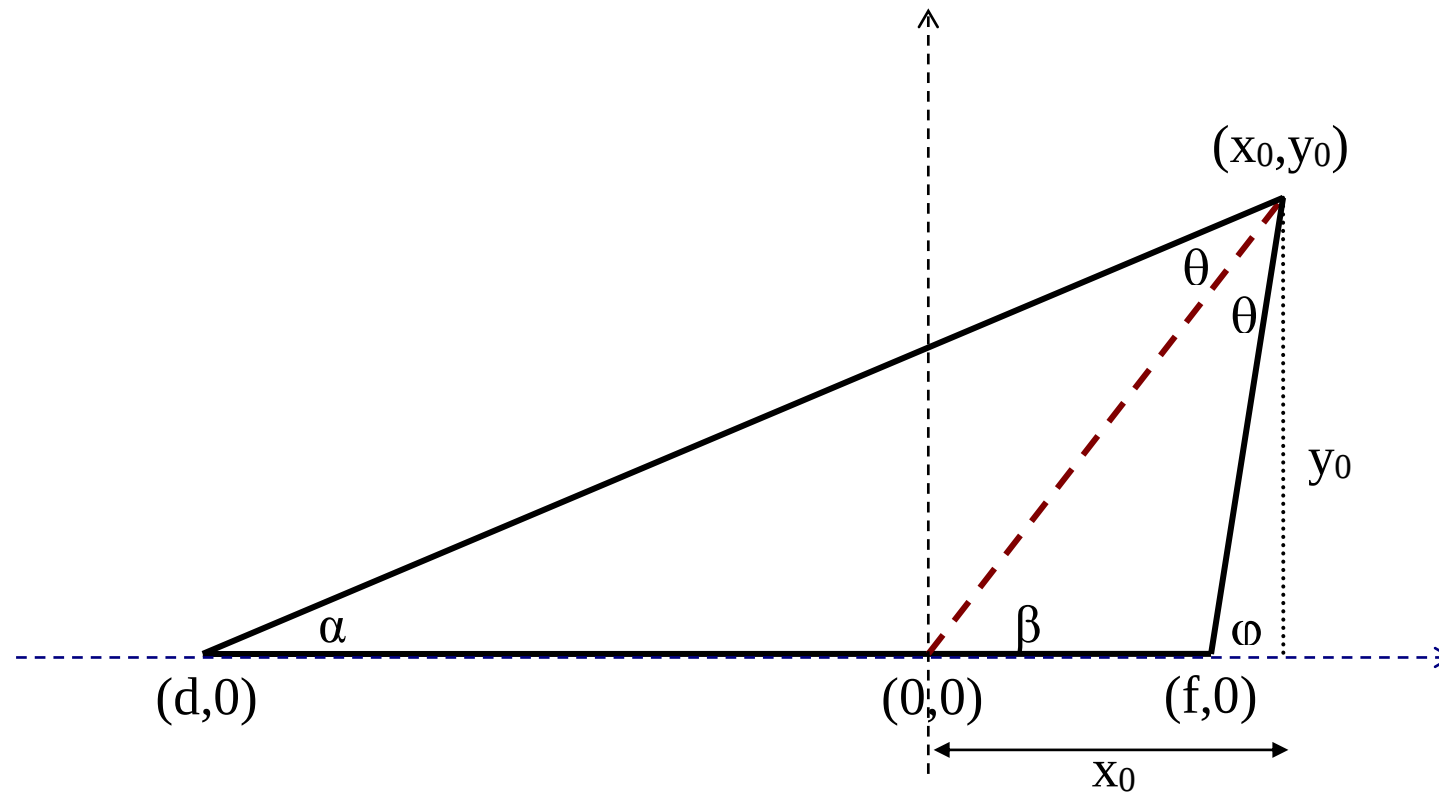




The triangle in figure is typical **obtuse** one i.e. the exterior angle ϕ is less than 90° . Here the base (of the altitude y_0) is chosen to be on the x-axis. The bisector of the vertex angle (the red dashed line) intersects with the base at the origin point $(0,0)$ of the Cartesian x-y coordinates. It's obvious that the tangents of the angles ϕ , β and α have the same numerator y_0 . This is because:

$\tan(\beta) = y_0 / x_0$, $\tan(\varphi) = y_0 / (x_0 - f)$ and $\tan(\alpha) = y_0 / (x_0 - d)$, **d here is negative!**

The base of the triangle consists of two parts f and d and the task is proposed to find in general the relationship between these two parts. Here is the approach!

The exterior angles $\varphi = \beta + \theta$, while $\beta = \theta + \alpha$, which gives:

$2\beta = \varphi + \alpha$, and therefore $\tan(2\beta) = \tan(\varphi + \alpha)$, by applying the tangent of sum:

$$\frac{2\tan(\beta)}{1 - \tan^2(\beta)} = \frac{\tan(\varphi) + \tan(\alpha)}{1 - \tan(\varphi)\tan(\alpha)}$$

This equation can be simplified by multiplying the numerator of one side with the denominator of the other side which results another equation:

$$[2\tan(\beta)] [1 - \tan(\varphi)\tan(\alpha)] = [\tan(\varphi) + \tan(\alpha)] [(1 - \tan^2(\beta))]$$

Prior continue it is important to be very careful with the **+/-** signs. Now substitute the tangents with their values:

$$[2y_0 / x_0] [1 - (y_0^2 / ((x_0 - f)(x_0 - d)))] = [y_0 / (x_0 - f) + y_0 / (x_0 - d)] [1 - (y_0^2 / x_0^2)]$$

We can rewrite the above equation in order to recognize common values:

$$\frac{2y_0}{x_0} \left[1 - \frac{y_0^2}{(x_0 - f)(x_0 - d)} \right] = \left[\frac{y_0}{(x_0 - f)} + \frac{y_0}{(x_0 - d)} \right] \left[1 - \frac{y_0^2}{x_0^2} \right]$$

You can notice that y_0 is common numerator on both sides of the equation which can be eliminated by dividing by y_0 and we get:

$$\frac{2}{x_0} \left[1 - \frac{y_0^2}{(x_0 - f)(x_0 - d)} \right] = \left[\frac{1}{(x_0 - f)} + \frac{1}{(x_0 - d)} \right] \left[1 - \frac{y_0^2}{x_0^2} \right]$$

Further simplifying:

$$\frac{2}{x_0} \left[\frac{(x_0 - f)(x_0 - d) - y_0^2}{[(x_0 - f)(x_0 - d)]} \right] = \left[\frac{(x_0 - d) + (x_0 - f)}{[(x_0 - f)(x_0 - d)]} \right] \left[\frac{x_0^2 - y_0^2}{x_0^2} \right]$$

Multiplying both sides by $x_0 [(x_0 - f)(x_0 - d)]$ will result:

$$2 [x_0^2 - y_0^2 - (f + d)x_0 + fd] = [2x_0 - (f + d)] \left[\frac{x_0^2 - y_0^2}{x_0} \right]$$

$$2(x_0^2 - y_0^2)x_0 - 2(f + d)x_0^2 + 2fdx_0 = 2(x_0^2 - y_0^2)x_0 - (f + d)x_0^2 + (f + d)y_0^2$$

$$2fdx_0 = (f + d)x_0^2 + (f + d)y_0^2 = (f + d)(x_0^2 + y_0^2)$$

Surprisingly $(x_0^2 + y_0^2)$ = the square of the length of the bisector (the red dashed line) which I'll denote by the letter r , and the last equation will become more simpler:

$2fdx_0 = (f + d)r^2$ and can be rewritten:
 $2x_0 / r^2 = (f + d) / fd$, and finally:

$$2x_0 / r^2 = 1/f + 1/d$$

This means that task is achieved perfectly!!

Important note: I used to consider the sine and cosine functions as the normalized unit-less values of the x- and y-coordinates regardless of the value of the radius of any circle. The last equation can be modified slightly to be presented by normalized values instead:

$$2 x_0/r = 1/(f/r) + 1/(d/r)$$

$$2 X_0 = \frac{1}{F} + \frac{1}{D}$$